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q-DEFORMING THE SYNCHROTRON SHAPE FUNCTION

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We replace the usual integral in the shape function of the synchrotron spectrum by a Jackson (q-deformed) integral and write down the formulas required to calculate the Jackson first deformed form of the synchrotron shape function.

Synchrotron radiation[1], first observed in 1947, is an extremely important phenomenon in the realm of physics. It is a nonthermal (magnetobremsstrahlung) radiation pattern, which can be encountered in many cyclic accelerators and in much wider astrophysical contexts [2]. On the other hand, the interesting elaborations on the quantum deformed (basic) calculus are well established in the mathematical literature [3], and over the past years there has been much interest to apply the q-deformed techniques to physical phenomena and theories. The purpose of this work is to present a q-deformation of the synchrotron shape function.

The spectral intensity of the magnetobremsstrahlung in the synchrotron regime is proportional to the so-called shape function [1]

$$W_\omega \propto F\left(\frac{\omega}{\omega_m}\right) , \tag{1}$$

where ω_m is given in terms of the cyclotron radian frequency ω_c as $\omega_m = \omega_c\, \gamma^3$, and F is given by $F(\xi) = \frac{9\sqrt{3}}{8\pi}\xi \int_\xi^\infty K_{5/3}(z)dz$, where K is the MacDonald function of the quoted fractional order. The small and large asymptotic limits of the synchrotron shape function are as follows

$$F(\xi \ll 1) \approx 1.33\, \xi^{1/3} \tag{2}$$

and

$$F(\xi \gg 1) \approx 0.78\, \xi^{1/2} e^{-\xi} , \tag{3}$$

with a maximum (amount of radiation) to be found at the frequency $\frac{1}{3}\omega_m$.

At the beginning of the century, F.H. Jackson [4] introduced the so called q-integrals, which are currently known as Jackson's integrals. By definition

$$\int_0^z f(t)d_qt = z(1-q)\sum_{n=0}^{\infty} f(zq^n)q^n . \tag{4}$$

On the other hand, Thomae and Jackson defined a q-integral on $(0,\infty)$ by

$$\int_0^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n . \tag{5}$$

Thus, one gets

$$\int_z^\infty f(t)d_qt = \int_0^\infty f(t)d_qt - \int_0^z f(t)d_qt \tag{6}$$

or

$$\int_z^\infty f(t)d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n - z(1-q)\sum_{n=0}^{\infty} f(zq^n)q^n \ . \tag{7}$$

In the case of the synchrotron radiation we have to take f as the q-deformed K function. To get this function we can use any of the three basic Bessel J functions one can encounter in the mathematical literature, which are expressed in terms of the basic hypergeometric functions ${}_2\phi_1$ (for the first Jackson Bessel function $J^{(J1)}$), ${}_0\phi_1$ (for the second Jackson Bessel function $J^{(J2)}$), and ${}_1\phi_1$ (for the Hahn-Exton Bessel function $J^{(HE)}$), respectively.

Here, we shall use the first Jackson form of the q-deformed J, because it does not imply the deformation of the argument as the other two basic analogs do (see, e.g., [5]), i.e.,

$$J_\nu^{(J1)} = \frac{1}{(q;q)_\nu}\left(\frac{x}{2}\right)^\nu {}_2\phi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right) \ . \tag{8}$$

From the general definition of the basic hypergeometric series

$${}_r\phi_s(a_1,a_2,...,a_r;b_1,b_2,...,b_s;q,x) = \sum_{n=0}^{\infty}\frac{(a_1;q)_n...(a_r;q)_n}{(b_1;q)_n...(b_s;q)_n}[(-1)^n q^{n(n-1)/2}]^{1+s-r}\frac{x^n}{(q;q)_n} \tag{9}$$

where the q-shifted factorial symbol is defined as

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \ , \tag{10}$$

$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$, $0<q<1$, and for α a positive integer n, $(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1})$, the basic hypergeometric series in the rhs of Eq. (8) can be calculated explicitly as follows

$${}_2\phi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right) = \sum_{n=0}^{\infty}\left(\Big[\prod_{k=1}^{n}(1-q^k)\Big]^{-1}\Big[\prod_{k=0}^{n}(1-q^{\nu+1+k}\Big]^{-1}\left(\frac{-x^2}{4}\right)^n\right) \ . \tag{11}$$

According to Ismail [6] the modified q-Bessel function of the first kind reads

$$I_\nu^{(J1)}(x;q) = e^{-i\pi\nu/2}J_\nu^{(J1)}(ix;q) \ . \tag{12}$$

One can now use the well-known relation between $I_\nu(x)$ and $K_\nu(x)$ [7] to define basic MacDonald functions $K^{(J1)}$

$$K_\nu^{(J1)}(x) = \frac{\pi}{2\sin(\nu\pi)}\left[I_{-\nu}^{(J1)}(x) - I_\nu^{(J1)}(x)\right] \ , \tag{13}$$

(no deformation on the sine function!). Of course Eq. (13) can be applied to all three types of q-deformations.

Thus, Jackson's first q-analog of the synchrotron shape function reads

$$F^{(J1)}(\xi) = \frac{9\sqrt{3}}{8\pi}\xi\Big[(1-q)\sum_{n=-\infty}^{\infty} K_{5/3}^{(J1)}(q^n)q^n - \xi(1-q)\sum_{n=0}^{\infty} K_{5/3}^{(J1)}(\xi q^n)q^n\Big] \quad (14)$$

and all the formulas needed to calculate $K^{(J1)}$ have been collected herein. As $q \rightarrow 1^-$, $F^{(J1)}(\xi)$ goes to $F(\xi)$.

Such global types of deformation can be applied to other forms of magneto-bremsstrahlung as well, e.g., FEL ones.

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